Adiabatic quantum state transfer in non-uniform triple-quantum-dot system

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We introduce an adiabatic quantum state transfer scheme in a non-uniform coupled triple-quantum-dot system. By adiabatically varying the external gate voltage applied on the sender and receiver, the electron can be transferred between them with high fidelity. By numerically solving the master equation for a system with *always-on* interaction, it is indicated that the transfer fidelity depends on the ration between the peak voltage and the maximum coupling constants. The effect of coupling mismatch on the transfer fidelity is also investigated and it is shown that there is a relatively large tolerance range to permit high fidelity quantum state transfer.

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I. INTRODUCTION

In quantum information science, quantum state transfer (QST), as the name suggests, refers to the transfer of an arbitrary quantum state from one qubit to another. Recently, there are two major mechanisms for QST. The first approaches are usually characterized by preparing the quantum channel with an *always-on* interaction where QST is equivalent to the time evolution of the quantum state in data bus [1–3]. However, these approaches require precise control of distance and timing. Any deviation may leads to significant errors. The other approaches have paid much attention to adiabatic passage for coherent QST in time-evolving quantum systems. The most well known example of these is the so-called Stimulated Raman Adiabatic Passage (STIRAP) technique, which is used to produce a complete population transfer between two internal quantum states of an atom or molecule [4]. Such methods are relatively insensitive to gate errors and other external noises and do not require an accurate control of the system parameters, thus can realize high-fidelity QST.

Due to the potential scalability and long decoherence times, the applications of adiabatic passage have been widely investigated in solid-state systems [5–8, 10–17]. Eckert et al. [7] have introduced an implementation of the STIRAP in the three-trap potential array. By coherently manipulating the trap separation between each two traps, the neutral atoms can be transferred in the millisecond range. Zhang et al. [8] have describe a scheme for using an all-electrical, adiabatic population transfer between two spatially separated dots in a triple-quantum-dot (TQD) system by adiabatically engineering external gate voltage. In ref. [10], A. D. Greentree et al. have described a method of coherent electronic transport through a triple-well system by adiabatically following a particular energy eigenstate of the system. By adiabatically modulating coherent tunneling rates between nearest neighbor dots, it can realize a high fidelity transfer. This method was termed Coherent Tunneling by Adiabatic Passage (CTAP) which was demonstrated experimentally very recently via optical waveguide [9]. Since then, adiabatic passage has also been used to transport quantum information from a single sender to multiple receivers, which relates to a quantum wire or fan-out [12]. Following a different perspective, there have been several recent proposals to coherently manipulate BECs [13–15] in triple-well potentials. Ref. [16] has analytically derived the condition for coherent tunneling via adiabatic passage in a triple-well system with negligible central-well population at all times during the transfer.

In CTAP technique [10], the basic idea is to use the existence of a spatial dark state which is a coherent superposition state of two "distant" spatial trapping sites,

$$|D_0\rangle = \cos\theta_1 |L\rangle + 0 |M\rangle - \sin\theta_1 |R\rangle ,$$

where the mixing angle θ_1 is defined as $\tan\theta_1 = \Omega^{LM}/\Omega^{MR}$ with Ω^{LM} (Ω^{MR}) denoting the tunneling rate between nearest-neighbor dots. By maintaining the system in state $|D_0\rangle$ and adiabatically manipulating the tunneling rates, it is possible to achieve coherent population transfer from site $|L\rangle$ to $|R\rangle$ without any probability being in the state $|M\rangle$. In this paper we consider a different adiabatic protocol to achieve population transfer between two spatially separated dots. We introduce a non-uniform coupled triple-quantum-dot array which can be manipulated by external gate voltage applied on the two external dots (sender and receiver). Through maintaining the system in the ground state we show that the electron initially in the left dot can be transferred to the right dot occupation with high fidelity. Furthermore, we study in details the dynamic competition between the adiabatic QST and the decoherence. There are two time scales T_1 and T_2 depicting such competition, where T_1 represents the adiabatic time limited by the adiabatic conditions and T_2 represents the decoherence time.

The paper is organized as follows. In Sec. II we setup the model and we describe the adiabatic transfer of an electron between quantum dots. We also derive a perturbative, analytical expression of fidelity. In Sec. III we show numerical results that substantiate the analytical results. The last section is the summary and discussion of this paper.

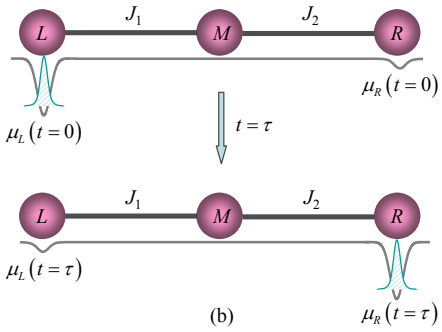

FIG. 1: (Color online) Schematic illustrations of quantum state transfer in non-uniform triple-dot system: (a) the system is controlled by gates voltage $\mu_\alpha(t)$ ($\alpha = L,\ R$); (b) by adiabatically change the $\mu_\alpha(t)$ ($\alpha = L,\ R$) one can achieve QST from $|L\rangle$ to $|R\rangle$.

II. MODEL SETUP

In this section, we first introduce the isolated (no coupling to the leads) TQD array $|L,\sigma\rangle$, $|M,\sigma\rangle$, $|R,\sigma\rangle$ ($\sigma = \uparrow,\downarrow$), where $|L,\sigma\rangle = c^\dagger_{L,\sigma}|\text{vac}\rangle$ ($|M\rangle = c^\dagger_{M,\sigma}|\text{vac}\rangle$, $|R\rangle = c^\dagger_{R,\sigma}|\text{vac}\rangle$) corresponds to an electron in the left (center, right) dot with spin σ. The scheme is schematically shown in Fig. 1(a). Specifically, we consider the interactions between nearest-neighbor quantum dots are timeless and slightly different. We term this model non-uniform triple-quantum-dot (NUTQD) system. The system are controlled by external time-varying gates voltage $\mu_\alpha(t)$ ($\alpha = L,\ R$), which control the site energies of two end of the array. In this proposal we will show that the information encoded in electronic spin can be transported from $\cos\theta\,|L,\uparrow\rangle + \sin\theta\,|L,\downarrow\rangle$ to $\cos\theta\,|R,\uparrow\rangle + \sin\theta\,|R,\downarrow\rangle$. Notice that the polarization of the spin of an electron is not changed as time evolves. Then the problem about the quantum information transfer (QIT) can be reduced to the issue of QST and a complete QST can achieve perfect QIT. In this sense, we can ignore spin degrees of freedom to illustrate the principles of QST from $|L\rangle$ to $|R\rangle$.

We use $\{|L\rangle, |M\rangle, |R\rangle\}$ as basis of the Hilbert space, the Hamiltonian for NUTQD system in matrix form reads as

$$H = \begin{bmatrix} \mu_L(t) & J_1 & 0 \\ J_1 & 0 & J_2 \\ 0 & J_2 & \mu_R(t) \end{bmatrix}, \tag{1}$$

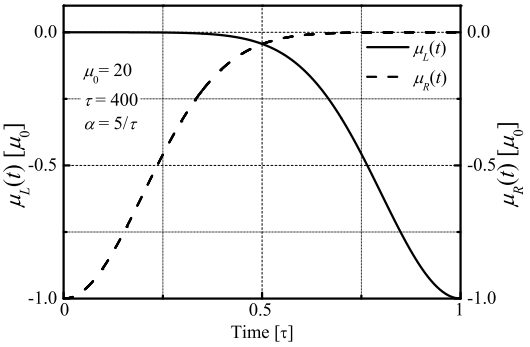

FIG. 2: Gate voltages as a function of time (in units of τ) , $\mu_L(t)$ is the solid line and $\mu_R(t)$ is the dash line.

where J_i ($i = 1, 2$) is the fixed coupling constant between nearest-neighbor dots, assumed to be real (negative) for the sake of simplicity. The on-site energies $\mu_L(t)$ and $\mu_R(t)$ are modulated in Gaussian pulses to realize the adiabatic transfer, according to (shown in Fig. 2)

$$\mu_L(t) = -\mu_L^{\max}\exp\left[-\frac{1}{2}\alpha^2 t^2\right], \tag{2a}$$

$$\mu_R(t) = -\mu_R^{\max}\exp\left[-\frac{1}{2}\alpha^2\,(t-\tau)^2\right], \tag{2b}$$

where τ and α are the total adiabatic evolution time and standard deviation of the control pulse modulating the chemical potential of states $|L\rangle$ and $|R\rangle$. For simplicity we set the peak voltage of each dot to be equal $\mu_L^{\max} = \mu_R^{\max} = \mu_0$ and satisfy $\mu_0 \gg |J_i|$ ($i = 1, 2$). We will see below that this simplicity has no relevance to the problem.

At time $t = t_0$, the Hamiltonian $H(t_0)$ has eigenvectors $|\psi_n(t_0)\rangle$ ($n = 0, 1, 2$) which are superpositions of $|L\rangle, |M\rangle, |R\rangle$ and the eigenvalues are denoted by $\varepsilon_n(t_0)$, sorting in ascending order $\varepsilon_0 < \varepsilon_1 < \varepsilon_2$. Under adiabatic evolution, these eigenstates evolve continuously to $|\psi_n(t)\rangle$. The instantaneous Hamiltonian's eigen equation is

$$H(t)\,|\psi_n(t)\rangle = \varepsilon_n(t)\,|\psi_n(t)\rangle\,. \tag{3}$$

In this proposal, we use ground state $|\psi_0(t)\rangle$ of Eq. (3) to induce population transfer from state $|L\rangle$ to $|R\rangle$ (see Fig. 1(b)). One advantage of this proposal is that there can be no heat dissipation during the transfer.

Starting from $t = 0$, the Hamiltonian is approximate separable in the case $\mu_0 \gg |J_i|$:

$$H(t=0) \simeq H_L \oplus H_{MR}, \tag{4}$$

with

$$H_L = -\mu_0\,|L\rangle\langle L|\,, \tag{5a}$$

$$H_{MR} = J_2\,(|M\rangle\langle R| + |R\rangle\langle M|)\,. \tag{5b}$$

This Hamiltonian has the eigenstates

$$\begin{aligned} |\psi_{\pm}(t=0)\rangle &= \frac{1}{\sqrt{2}}(|M\rangle \pm |R\rangle), \\ |\psi_0(t=0)\rangle &= |L\rangle, \end{aligned} \tag{6}$$

the energies of these states are

$$\varepsilon_{\pm} = \pm J_2,\ \varepsilon_0 = -\mu_0. \tag{7}$$

Our aim is to induce population transfer from state $|L\rangle$ to $|R\rangle$ by maintaining the system in ground state. Now we will show that an adiabatic change of $\mu_L(t)$ and $\mu_R(t)$ will realize the QST.

In the adiabatic limit, $t \to \tau$, the parameter $\mu_L(t)$ goes to zero and $\mu_R(t)$ goes to $-\mu_0$. The Hamiltonian adiabatically evolves to

$$H(t=\tau) \simeq H_{LM} \oplus H_R, \tag{8}$$

with

$$H_{LM} = J_1(|L\rangle\langle M| + |M\rangle\langle L|), \tag{9a}$$
$$H_R = -\mu_0 |R\rangle\langle R|, \tag{9b}$$

the corresponding eigenstate are

$$\begin{aligned} |\psi_{\pm}(t=\tau)\rangle &= \frac{1}{\sqrt{2}}(|L\rangle \pm |M\rangle), \\ |\psi_0(t=\tau)\rangle &= |R\rangle. \end{aligned} \tag{10}$$

and then the ground state evolves to be $|R\rangle$.

Providing adiabaticity is satisfied [19]

$$|\varepsilon_m - \varepsilon_n| \gg |\langle\psi_m|\dot{\psi}_n\rangle|, \tag{11}$$

the overall system will remain in its instantaneous ground state. At $t = 0$, the system is prepared in state $|\psi_0(t=0)\rangle = |L\rangle$, then the adiabatic theorem states that the system will stay in $|\psi_0(t)\rangle$. Note that $|L\rangle$ and $|R\rangle$ denote the states in which the electron is on the left and right QD, respectively. Therefore, we can see that an electron starting in $|L\rangle$ will end up in $|R\rangle$.

Providing the length of time τ is too large, that is, the time-dependent change is introduced slowly enough, the fidelity of QST is also determined by peak gate voltage μ_0. Notice that the square of the module of fidelity $|F(t)|^2 = |\langle R|\psi_0(t)\rangle|^2$ denotes the probability of finding $|R\rangle$ in the ground state $|\psi_0(t)\rangle$. Now we suppose to get analytical expression of fidelity using first order perturbation theory. We start from Eq. (1) at $t = \tau$ and consider the coupling term $J_2(|R\rangle\langle M| + |M\rangle\langle R|)$ as a weak perturbation. The Hamiltonian

$$H(t=\tau) = H_0 + H_I, \tag{12}$$

contains two parts

$$H_0 = J_1(|L\rangle\langle M| + |M\rangle\langle L|) - \mu_0|R\rangle\langle R|, \tag{13a}$$
$$H_I = J_2(|R\rangle\langle M| + |M\rangle\langle R|). \tag{13b}$$

Our aim is to find the approximate expression for the ground state $|\psi_0\rangle$ of the perturbed Hamiltonian $H(t=\tau)$. The eigenfunctions of unperturbed Hamiltonian H_0 is

$$\begin{aligned} |\psi_0^{(0)}\rangle &= |R\rangle, \\ |\psi_{\pm}^{(0)}\rangle &= \frac{1}{\sqrt{2}}(|L\rangle \pm |M\rangle). \end{aligned} \tag{14}$$

In the picture of $\left\{|\psi_-^{(0)}\rangle, |\psi_+^{(0)}\rangle, |\psi_0^{(0)}\rangle\right\}$, The Hamiltonian H_0 can be diagonalized as

$$H_0 = \begin{bmatrix} -J_1 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & -\mu_0 \end{bmatrix}.$$

As the first order perturbation, we have the corrected ground state to be

$$\begin{aligned} |\psi_0\rangle &= |\psi_0^{(0)}\rangle + \sum_{\eta=\pm} \frac{\langle\psi_\eta^{(0)}|H_I|\psi_0^{(0)}\rangle}{E_0^{(0)} - E_\eta^{(0)}} |\psi_\eta^{(0)}\rangle \\ &= \frac{J_1 J_2}{\mu_0^2 - J_1^2}|L\rangle - \frac{\mu_0 J_2}{\mu_0^2 - J_1^2}|M\rangle + |R\rangle. \end{aligned} \tag{15}$$

So the transfer fidelity of adiabatic QST at $t = \tau$ is

$$\begin{aligned} |F(\tau)|^{-2} &= 1 + \left(\frac{J_1 J_2}{\mu_0^2 - J_1^2}\right)^2 + \left(\frac{\mu_0 J_2}{\mu_0^2 - J_1^2}\right)^2 \\ &= 1 + \frac{J_2^2(\mu_0^2 + J_1^2)}{(\mu_0^2 - J_1^2)^2}, \end{aligned} \tag{16}$$

which shows that the peak voltage μ_0 determined the fidelity of QST. As $\mu_0 \gg |J_i|$ is satisfied, the fidelity is near to unity.

III. NUMERICAL SIMULATIONS

The analysis above is based on the assumption that the adiabaticity is satisfied. In order to demonstrate the QST in the system (1) and to show how exact the approximation is, in this section we numerically solve the master equation and the above central conclusion can be get confirmed. The main goal of this section is to analyze the parameters which influence the fidelity of adiabatic QST and find the proper matching relation between them.

First, initialize electron in the left dot, i.e., the total initial state is $|\Psi(0)\rangle = |L\rangle$, the time evolution creates a coherent superposition:

$$|\Psi(t)\rangle = c_1(t)|L\rangle + c_2(t)|M\rangle + c_3(t)|R\rangle. \tag{17}$$

with this notation we assume the initial condition $c_1(0) = 1$, and the other two equal zero. In order to proceed, we numerically solve the master equations for the density matrix ρ. The master equation is written as [19] (assuming $\hbar = 1$)

$$i\frac{d\rho(t)}{dt} = [H, \rho(t)], \tag{18}$$

where $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$. With the basis state ordering $\{|L\rangle, |M\rangle, |R\rangle\}$, the density matrix can be written as

$$\rho(t) = \begin{bmatrix} |c_1(t)|^2 & c_1(t)c_2^*(t) & c_1(t)c_3^*(t) \\ c_2(t)c_1^*(t) & |c_2(t)|^2 & c_2(t)c_3^*(t) \\ c_3(t)c_1^*(t) & c_3(t)c_2^*(t) & |c_3(t)|^2 \end{bmatrix}.$$

According to the definition of fidelity, we can see that $|F(t)|^2 = |c_3(t)|^2$. The crucial requirement for adiabatic evolution is Eq. (11). Firstly, one must to make sure that no level crossings occur, i.e., $\varepsilon_0(t) - \varepsilon_j(t) < 0$. To calculate the energies is generally only possible numerically. In Fig. 3(a) we present the results showing the eigenenergy gap $\Delta(t) = \varepsilon_1(t) - \varepsilon_0(t)$ between the first-excited state and ground state of the NUTQD system undergoing evolution due to modulation of the gate voltage according to pulse Eq. (1) for $\mu_0 = 20$, $J_1 = 0.8$, $J_2 = 1.0$, $\tau = 10\mu_0/J_1^2$ and $\alpha = 3/\tau$, $4/\tau$, $5/\tau$, $6/\tau$. It shows that for the given evolution time $\tau = 400$ the minimum of the energy gap decrease as standard deviation α increasing. The slower Hamiltonian (1) varies, the closer adiabatic theorem holds. In Fig. 3 we also show the numerically computed behavior of the populations $|c_1(t)|^2$, $|c_2(t)|^2$ and $|c_3(t)|^2$ on the three quantum dots as a function of time with $\alpha = 4/\tau$ and $\alpha = 5/\tau$. Note that for $\alpha = 4/\tau$ transfer, as illustrated in Fig. 3(b), the population on state $|R\rangle$ is decoupled and stays constant 0.92. The fraction of population left in states $|L\rangle$ and $|M\rangle$ is $|c_1(\tau)|^2 + |c_2(\tau)|^2 = 0.08$ and executes Rabi oscillations because the quantum dots L and M are coupled with $J_1 = 0.8$. Whereas for $\alpha = 5/\tau$ case, shown in Fig. 3(c), one can see that the fidelity of adiabatic QST has been improved considerably by this slight change. The fidelity of QST achieve 0.995 and only 0.5% of population remains in states $|L\rangle$ and $|M\rangle$. This is consistent with the results shown in Fig. 3(a) because the eigenenergy gap plays opposite role for transition probability.

The fidelity of population transfer will be very high as long as the Hamiltonian evolves sufficiently slowly in time (as determined by criteria for the applicability of the theorem). In practice the maximum possible transfer rates will be a few times greater than μ_0/J_1^2 which is illustrated in Fig. 4. Note that the transfer fidelity becomes stable when the total evolution time satisfy $\tau \geq 4\mu_0/J_1^2$.

The preceding discussion is based on the assumption that the system parameters are setup with arbitrary precision that is the system is coupled with $J_1 = 0.8$ and

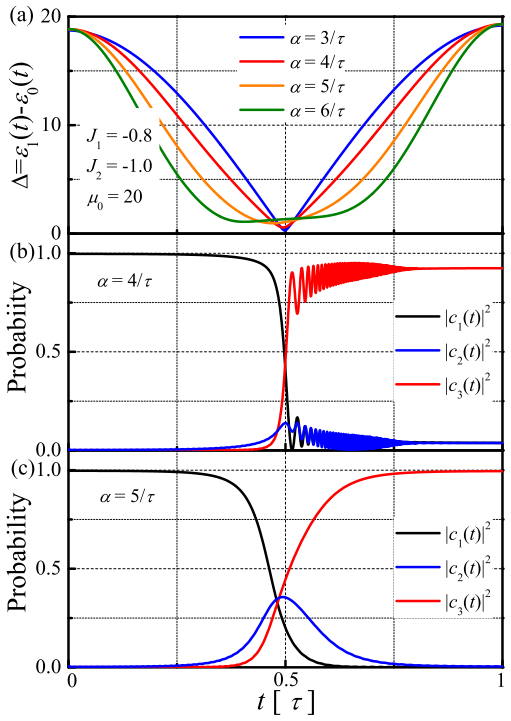

FIG. 3: (Color online) (a) The energy gap $\Delta(t) = \varepsilon_1(t) - \varepsilon_0(t)$ between the first-excited state and ground state of the triple-dot system undergoing evolution due to modulation of the gate volgate according to pulse Eq. (3) for $\mu_0 = 20$, $J_1 = 0.8$, $J_2 = 1.0$, $\tau = 10\mu_0/J_1^2$ and $\alpha = 3/\tau$, $4/\tau$, $5/\tau$, $6/\tau$. The time evolution of the probabilities induced by the pulses in Fig. 2 for (b) $\alpha = 4/\tau$ and (c) $\alpha = 5/\tau$. Initially the population is on left qubit (black line) and finally mainly on right qubit (red line). The population on the intermediate qubit is shown as a blue line.

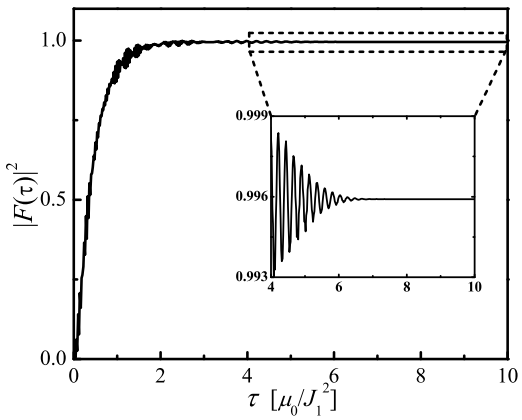

FIG. 4: Fidelity as a function of total adiabatic evolution time τ (in units of μ_0/J_1^2). When $\tau \geq 4\mu_0/J_1^2$, the fidelity of QST becomes stable.

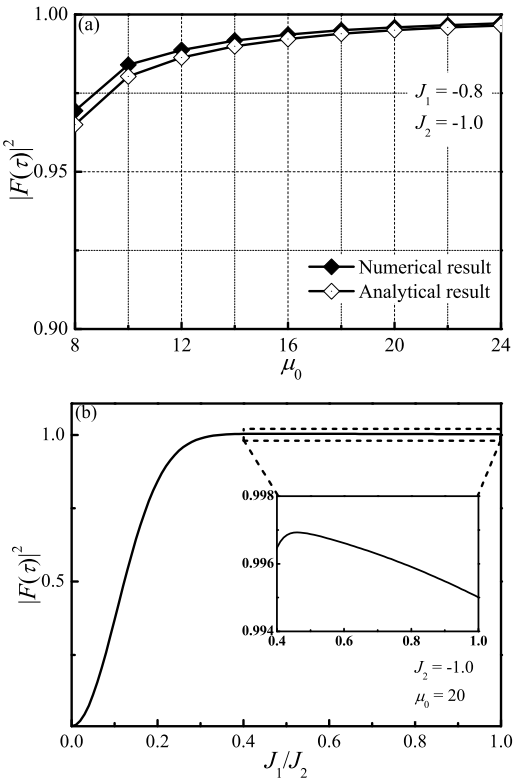

FIG. 5: The plot of the square of fidelity $|F(\tau)|^2$ as a function of system paremeters: (a) the peak voltage μ_0 and (b) the ratio J_1/J_2. If the condition is satisfied when $|\mu_0/J_{max}| \geq 14$ and $J_1/J_2 \geq 0.4$, the transfer fidelity is near to one.

$J_2 = 1.0$. However, it is difficult to fabricate such precise Hamiltonian in experiment. Next we will show that the adiabatic passage like us is relatively insensitive to the system parameters. From the analytical results, the fidelity of adiabatic QST depends on the contrast ratio between peak voltage μ_0 and coupling constants J_i. To determine the parameter range needed to achieve high fidelity transfer, we numerically integrate the density matrix equations of motion, with varying the peak voltage μ_0. In Fig. 5(a) we present results showing the square of fidelity $|F(\tau)|^2 = |c_3(\tau)|^2$ as a function of μ_0 with $J_1 = 0.8$, $J_2 = 1.0$, $\tau = 375$ and $\alpha = 5/\tau$. We can see that the population transfer is close to one ($|F(\tau)|^2 \geq 0.99$) and stable when μ_0 is achieved for $|\mu_0/J_2| \geq 14$. The plot in Fig. 5(a) is in agreement with the analytical results Eq. (16) with high accuracy. On the other hand, the difference between J_1 and J_2 has a little effect upon transfer fidelity within certain range. We have illustrated this in Fig. 5(b) where the effects of mismatch between J_1 and J_2 have been modeled. Here we show $|F(\tau)|^2$ as a function of J_1/J_2 for peak voltage $\mu_0 = 20$ to simulate the effect of a systematic error in the coupling constants. Note that the ratio as much as 0.35 still permits $|F(\tau)|^2 \approx 0.994$.

IV. SUMMARY AND DISCUSSION

In summary, we have introduced a method of coherent QST through a NUTQD system by adiabatic passage. This scheme is realized by modulation of gate voltage of QDs. Different from the CTAP Scheme, our method is to induce population transfer by maintaining the system in its ground state which is more stable than dark state. We have studied the adiabatic QST through a NTQD system by theoretical analysis and numerical simulations of the ground state evolution of NTQD model. The result shows that it is a high fidelity process for a proper choose of standard deviation and peak voltage.

In order to investigate the relation between the fidelity of quantum state transfer $|F(\tau)|^2$ and peak voltage μ_0, we have numerically solve the master equation under different peak voltage. The numerical result shows that if we want to achieve a high fidelity more than 99.5% we require the ratio of $|\mu_0/J_2| \geq 14$. We also show that the sight difference between J_1 and J_2 does small influence on the fidelity.

It is worthwhile to discuss the applicability of the scheme presented above. In a real system, quantum decoherence is the main obstacle to the experimental implementation of quantum information. For coupled QDs, experiments [20] show that the coupling strength J is about 0.25 meV while $\mu_0 \sim 20J$. we can estimate a time of ~ 50 ps required for adiabatic operation. On the other hand, the typical decoherence time T_2 for electron-spin has been indicated experimentally [21] to be longer than 80 ± 9 μs at 2.5 K which is much longer than adiabatic operation time. So our scheme has applicability in practice.

Acknowledgments

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